

Partitions of groups into large subsets

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Abstract

Let G be a group and let κ be a cardinal. A subset A of G is called left (right) κ -large if there exists a subset F of G such that $|F| < \kappa$ and $G = FA$ ($G = AF$). We say that A is κ -large if A is left and right κ -large. It is known that every infinite group G can be partitioned into countably many \aleph_0 -large subsets. On the other hand, every amenable (in particular Abelian) group G cannot be partitioned into $> \aleph_0$ \aleph_0 -large subsets.

We prove that every infinite group G of cardinality κ can be partitioned into κ left- \aleph_1 -large subsets and every free group F_κ in the infinite alphabet κ can be partitioned into κ 4-large subsets.

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1 Introduction

Let G be a group, κ be a cardinal, $[G]^{<\kappa}$ denotes the family of all subsets of G of cardinality $< \kappa$. A subset A of G is called

- *left (right) κ -large* if there exists $F \in [G]^{<\kappa}$ such that $G = FA$ ($G = AF$);
- *κ -large* if A is left and right κ -large.

We note that A is left κ -large if and only if A^{-1} is right κ -large. In the dynamical terminology [3, 3, p.101], the left \aleph_0 -large subsets are known under the name *syndetic subsets*.

In [2], Bella and Malykhin asked whether every infinite group G can be partitioned into two \aleph_0 -large subsets. This question was answered in [5] (see also [8, Theorem 3.12]): G can be partitioned into countably many \aleph_0 -large subsets.

If G is amenable (in particular, G is Abelian) and μ is a left invariant Banach measure on G then $\mu(A) > 0$ for every left \aleph_0 -large subset A of G . It follows that G cannot be partitioned into $> \aleph_0$ left \aleph_0 -large subsets.

On the other hand (see [7, Theorem 2.4] or [8, Theorem 12.11]), every infinite group G can be partitioned into κ left κ -large subsets for each infinite cardinal $\kappa \leq |G|$. In this connection, we asked [8, Question 12.6] whether an infinite Abelian group G of cardinality \aleph_2 can be partitioned into \aleph_2 \aleph_1 -large subsets.

In section 2 of this paper, we prove that every infinite group G of cardinality κ can be partitioned into κ left \aleph_1 -large subsets.

In section 3 we partition the free group F_κ in the infinite alphabet κ into κ left 3-large subsets and into κ 4-large subsets.

In section 4 we consider two alternative examples of partitions of G -spaces into large subsets and conclude the paper with some comments and open problems in Section 5.

2 Partitions and filtrations

Let G be an infinite group with the identity e , \aleph be an infinite cardinal. A family $\{G_\alpha : \alpha < \aleph\}$ of subgroups of G is called a *filtration* if the following conditions hold

- (1) $G_0 = \{e\}$ and $G = \bigcup_{\alpha < \aleph} G_\alpha$;
- (2) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < \aleph$;
- (3) $\bigcup_{\alpha < \beta} G_\alpha = G_\beta$ for each limit ordinal $\beta < \aleph$.

Clearly, a countable group G admits a filtration if and only if G is not finitely generated. Every uncountable group G of cardinality \aleph admits a filtration satisfying the additional condition $|G_\alpha| < \aleph$ for each $\alpha < \aleph$.

Following [6], for each $0 < \alpha < \aleph$, we decompose $G_{\alpha+1} \setminus G_\alpha$ into right cosets by G_α and choose some system X_α of representatives so $G_{\alpha+1} \setminus G_\alpha = G_\alpha X_\alpha$. We take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup G_α with $g \in G_\alpha$. By (3), $\alpha = \alpha_1 + 1$ for some ordinal $\alpha_1 < \aleph$. Hence, $g \in G_{\alpha_1+1} \setminus G_{\alpha_1}$ and there exist $g_1 \in G_{\alpha_1}$ and $x_{\alpha_1} \in X_{\alpha_1}$ such that $g = g_1 x_{\alpha_1}$. If $g_1 \neq e$, we choose the ordinal α_2 and elements $g_2 \in G_{\alpha_2+1} \setminus G_{\alpha_2}$ and $x_{\alpha_2} \in X_{\alpha_2}$ such that $g_1 = g_2 x_{\alpha_2}$. Since the set of ordinals $\{\alpha : \alpha < \aleph\}$ is well-ordered, after finite number $s(g)$ of steps, we get the representation

$$g = x_{\alpha_{s(g)}} x_{\alpha_{s(g)-1}} \dots x_{\alpha_2} x_{\alpha_1}, \quad x_{\alpha_i} \in X_{\alpha_i}.$$

We note that this representation is unique and put

$$\gamma_1(g) = \alpha_1, \quad \gamma_2(g) = \alpha_2, \quad \dots, \quad \gamma_{s(g)}(g) = \alpha_{s(g)}.$$

Theorem 2.1. *Every infinite group G of cardinality \aleph can be partitioned into \aleph left \aleph_1 -large subsets.*

Proof. If G is countable, the statement is evident because each singleton is \aleph_1 -large. Assume that $\aleph > \aleph_0$ and fix some filtration $\{G_\alpha : \alpha < \aleph\}$ of G such that $|G_1| = \aleph_0$. Given any $g \in G \setminus \{e\}$, we rewrite the canonical representation $g = x_{\gamma_m} \dots x_{\gamma_1}$, in the following

$$g = g_1 x_{\gamma_m} \dots x_{\gamma_1},$$

$g_1 \in G_1$, $0 < \gamma_m < \dots < \gamma_1$. Here $g_1 = e$ and $m = n$ if $\gamma_n > 0$, and $g_1 = x_{\gamma_n}$ and $m = n - 1$ if $\gamma_n = 0$. We put $\Gamma(g) = \{\gamma_1, \dots, \gamma_m\}$ and fix an arbitrary bijection $\pi : G_1 \rightarrow \mathbb{N}$.

We define a family $\{A_\alpha : 0 < \alpha < \aleph\}$ of subsets of G by the following rule: $g \in A_\alpha$ if and only if $\alpha \in \Gamma(g)$ and $\gamma_{\pi(g_1)} = \alpha$. Since the subsets $\{A_\alpha : 0 < \alpha < \aleph\}$ are pairwise disjoint, it suffices to show that each A_α is left \aleph_1 -large. We take $a_\alpha \in X_\alpha$, put $F_\alpha = \{e, a_\alpha\}G_1$ and proof that $G = F_\alpha A_\alpha$.

Let $g \in G$ and $\alpha \in \Gamma(g)$. By the definition of A_α , there exists $h \in G_1$ such that $hg \in A_\alpha$ so $g \in G_1 A_\alpha$. If $\alpha \notin \Gamma(g)$ then $\alpha \in \Gamma(a_\alpha^{-1}g)$ so $a_\alpha^{-1}g \in G_1 A_\alpha$ and $g \in a_\alpha G_1 A_\alpha$. \square

3 Partitions of free groups

For a cardinal \aleph , we denote by F_\aleph the free group in the alphabet \aleph . Given any $g \in F_\aleph \setminus \{e\}$ and $a \in \aleph$, we write $\lambda(g) = a$ ($\rho(g) = a$) if the first (the last) letter in the canonical representation of g is either a or a^{-1} .

Lemma 3.1. *Suppose that a group G is a quotient of a group H , $f : H \rightarrow G$ is a quotient mapping. If \mathcal{P} is a partition of G into \aleph λ -large subsets then $\{f^{-1}(P) : P \in \mathcal{P}\}$ is a partition of H into \aleph λ -large subsets.*

Proof. For each $g \in G$, we choose some element $h_g \in f^{-1}(g)$. If $G = XP$ or $G = PX$ then $H = \{h_x : x \in X\}f^{-1}(P)$ or $H = f^{-1}(P)\{h_x : x \in X\}$. \square

Theorem 3.1. *For any infinite cardinal κ , the following statements hold*

- (i) F_κ can be partitioned into κ left 3-large subsets;
- (ii) F_κ can be partitioned into κ 4-large subsets.

Proof. (i) For each $a \in \kappa$, we put $P_a = \{g \in F_\kappa \setminus \{e\} : \lambda(g) = a\}$ and note that $F_\kappa = \{e, a\}P_a$.

(ii) We partition κ into 2-element subsets $\kappa = \bigcup_{\alpha < \kappa} \{x_\alpha, y_\alpha\}$ and put $X = \{x_\alpha : \alpha < \kappa\}$, $Y = \{y_\alpha : \alpha < \kappa\}$.

For every $\alpha < \kappa$, we denote

$$L_\alpha = \{g \in F_\kappa \setminus \{e\} : \lambda(g) = x_\alpha \Leftrightarrow \rho(g) \in X, \lambda(g) = y_\alpha \Leftrightarrow \rho(g) \in Y, \}$$

$$R_\alpha = \{g \in F_\kappa \setminus \{e\} : \rho(g) = x_\alpha \Leftrightarrow \lambda(g) \in Y, \rho(g) = y_\alpha \Leftrightarrow \lambda(g) \in X, \}$$

Then we put $P_\alpha = L_\alpha \cup R_\alpha$ and note that the subsets $P_\alpha : \alpha < \kappa$ are pairwise disjoint.

Given any $g \in F_\kappa$, we have

$$\{e, x_\alpha, y_\alpha\}g \cap L_\alpha \neq \emptyset, \quad g\{e, x_\alpha, y_\alpha\} \cap R_\alpha \neq \emptyset.$$

Hence, L_α is left 4-large and R_α is right 4-large, so P_α is 4-large. \square

Remark 3.1. Let G be a group, X be a left 3-large subset of G , Y be a right 3-large subset of G and $X \cap Y = \emptyset$. We show that $G = X \cup Y$. In particular each group can be partitioned in at most 2 3-large subsets.

Assume the contrary, take any $g \in G \setminus (X \cup Y)$. By the assumptions $G = \{e, x\}X = Y\{e, y\}$ for some $x, y \in G$. Then $x^{-1}g \in X$, $gy^{-1} \in Y$ so $x^{-1}gy^{-1} \in X$ and $x^{-1}gy^{-1} \in Y$ contradicting $X \cap Y = \emptyset$.

Theorem 3.2. *For a natural number $n \geq 2$, the following statements hold*

- F_n can be partitioned into \aleph_0 left 3-large subsets;
- F_n can be partitioned into \aleph_0 5-large subsets.

Proof. Given any $g \in F_n \setminus \{e\}$, $a \in n$ and $m \in \mathbb{N}$, we write $\bar{\lambda}(g) = a^m$ ($\bar{\rho}(g) = a^m$) if $g = a^{\pm m}h$, $\lambda(h) \neq a$ ($g = ha^{\pm m}$, $\rho(h) \neq a$).

- (i) We fix two distinct letters a, b from n and, for each $m \in \mathbb{N}$, put

$$P_m = \{g \in F_n \setminus \{e\} : \bar{\lambda}(g) = a^m\}$$

Clearly, $\{a^m, a^mb\}g \cap P_m \neq \emptyset$ for each $g \in F_n$, so P_m is left 3-large.

(ii) We suppose that $n = 2$ and F_2 is a free group in the alphabet $\{a, b\}$. For every $m \in \mathbb{N}$ we denote

$$L_m = \{g \in F_2 \setminus \{e\} : \bar{\lambda}(g) = a^m \Leftrightarrow \bar{\rho}(g) = a, \bar{\lambda}(g) = b^m \Leftrightarrow \bar{\rho}(g) = b\},$$

$$R_m = \{g \in F_2 \setminus \{e\} : \bar{\rho}(g) = a^m \Leftrightarrow \bar{\lambda}(g) = b, \bar{\rho}(g) = b^m \Leftrightarrow \bar{\lambda}(g) = a\}.$$

Then we put $P_m = L_m \cup R_m$ and note that the subsets $\{P_m : m \in \mathbb{N}\}$ are pairwise disjoint.

Given any $g \in F_2$, we have

$$\{a^m, b^m, a^mb, b^ma\}g \cap L_m \neq \emptyset, \quad g\{a^m, b^m, ba^m, ab^m\} \cap R_m \neq \emptyset.$$

Hence, L_m is left 5-large, R_m is right 5-large, so P_m is 5-large.

If $n > 2$ then F_2 is a quotient of F_n and we can apply Lemma 3.1. \square

Remark 3.2. For every $\kappa > 1$, one can easily choose a disjoint family $\{x_n H_n : n \in \omega\}$ of cosets of F_κ by some subgroup of finite index. Let G be an arbitrary group, H be a subgroup of finite index, $a \in G$, $b \in G$. The set aHb is called a *shifted subgroup* of finite index. Clearly, each shifted subgroup of finite index is \aleph_0 -large. We show that $|\mathcal{P}| \leq \aleph_0$ for any family \mathcal{P} of pairwise disjoint shifted subgroups of G of finite index. We denote by N the intersection of all normal subgroups of finite index of G , and let $f : G \rightarrow G/N$ denotes the quotient mapping. Let H be the profinite completion of G/N and let μ be the Haar measure on H . We observe that the subsets $\{cl_H f(P) : P \in \mathcal{P}\}$ are pairwise disjoint and $\mu(cl_H f(P)) > 0$ for each $P \in \mathcal{P}$. It follows that $|\mathcal{P}| \leq \aleph_0$.

4 On partitions of G -spaces

Let G be a group and let X be a transitive G -space with the action $G \times X \rightarrow X$, $(g, x) \mapsto gx$. For a cardinal λ , a subset Y of X is called λ -large if there exists $F \in [G]^{<\lambda}$ such that $X = FY$.

Example 4.1. Let κ be an infinite cardinal, $X = \kappa$ and let G denotes the group of all permutations of X . Let Y be a subset of X such that $|Y| = |X \setminus Y|$. We take a permutation $f \in G$ such that $f(Y) = X \setminus Y$, $f(X \setminus Y) = Y$. Then $X = \{e, f\}Y$. It follows that X can be partitioned into κ 3-large subsets.

Example 4.2. Let κ be an infinite cardinal, $X = \kappa$ and let G denotes the group of all permutations of X with finite support. Let Y be a subset of G such that $|X \setminus Y| = \kappa$. We take an arbitrary $F \in [G]^{<\kappa}$ and note that $(X \setminus Y) \setminus FY \neq \emptyset$. It follows that Y is not κ -large. Hence, X cannot be partitioned even into two κ -large subsets.

5 Comments and open questions

1. We begin with two open question arising naturally from the results of the paper.

Question 5.1. *Can every infinite group G of cardinality κ be partitioned into κ \aleph_1 -large subsets?*

Question 5.2. *For an infinite cardinal κ , can the free group F_κ be partitioned into κ 3-large subsets?*

Question 5.3. *For a natural number $n \geq 2$, can the free group F_n be partitioned into \aleph_0 4-large subsets?*

2. We do not know [7, Problem 4.2] whether every infinite group G of cardinality κ can be partitioned into κ κ -large subsets.

3. For every $n \in \mathbb{N}$, there is a (minimal) natural number $\Phi(n)$ such that, for every group G and every partition $G = A_1 \cup \dots \cup A_n$ there exists a cell A_i and a finite subset F of G such that $G = FA_i A_i^{-1}$ and $|F| \leq \Phi(n)$. So far it is an open problem [4, Problem 13.44] whether $\Phi(n) = n$. For nowadays state of this problem see the survey [1].

4. In [9] we conjectured that every infinite group G of cardinality κ can be partitioned $G = \bigcup_{n \in \omega} A_n$ such that each subset $A_n A_n^{-1}$ is not left κ -large. We confirmed this conjecture for each group of regular cardinality and of some groups (in particular, Abelian) of an arbitrary cardinality.

5. Every infinite group G of regular cardinality κ can be partitioned $G = A_1 \cup A_2$ such that A_1 and A_2 are not left κ -large. In [10] we show that this statement fails to be true for every Abelian group of singular cardinality κ .

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